

A note on global regularity in optimal transportation

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Abstract We indicate how recent work of Figalli–Kim–McCann and Vetois can be used to improve previous results of Trudinger and Wang on classical solvability of the second boundary value problem for Monge–Ampère type partial differential equations arising in optimal transportation together with the global regularity of the associated optimal mappings.

Keywords Optimal transportation · Monge–Ampère equations ·
Second boundary value problem

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1 Introduction

In this short note we use recent work of Figalli–Kim–McCann [4] and Vetois [24] on continuous differentiability and strict convexity of potentials in optimal transportation to improve previous results on global regularity of optimal mappings in [20]. The corresponding partial differential equations are Monge–Ampère type equations which have the general form:

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$$\det[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du), \quad (1.1)$$

where A and B are given $n \times n$ matrix and scalar valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, where Ω is a domain in Euclidean n -space, \mathbb{R}^n . We use (x, z, p) to denote points in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ so that $A(x, z, p) \in \mathbb{R}^n \times \mathbb{R}^n$, $B(x, z, p) \in \mathbb{R}$. Equation (1.1) is elliptic with respect to a solution $u \in C^2(\Omega)$ whenever

$$D^2u - A(\cdot, u, Du) > 0, \quad (1.2)$$

whence also $B > 0$.

We assume that A is generated by a *cost function* $c \in C^4(\bar{\Omega} \times \bar{\Omega}^*)$ satisfying the following conditions, where $\Omega^* \subset \mathbb{R}^n$ is a target domain:

A1: The mappings $c_x(x, \cdot)$, $c_y(\cdot, y)$ are one-to-one for each $x \in \Omega$, $y \in \Omega^*$;

A2: $\det c_{x,y} \neq 0$ in $\bar{\Omega} \times \bar{\Omega}^*$.

Conditions A1, A2 imply the existence of a C^3 mapping Y given by $Y(x, p) = c_x(x, \cdot)^{-1}(p)$ for $x \in \Omega$ and $p \in c_x(x, \cdot)(\Omega^*)$, with $Y_p = [c_{x,y}]^{-1}$. Moreover, for positive densities $f \in C^0(\Omega)$, $g \in C^0(\Omega^*)$, the *prescribed Jacobian equation*,

$$|\det DY(\cdot, Du)| = f/g^\circ Y \quad (1.3)$$

can be written in the form (1.1) with

$$A(\cdot, z, p) = c_{xx}(\cdot, Y(\cdot, p)), \quad B(x, z, p) = |\det c_{x,y}(\cdot, Y)|f/g^\circ Y, \quad (1.4)$$

(independent of z), for elliptic solutions u .

The *second boundary value problem* for the prescribed Jacobian equation (1.3) is to prescribe the image,

$$Tu(\Omega) := Y(\cdot, Du)(\Omega) = \Omega^*, \quad (1.5)$$

and a necessary condition for the existence of an elliptic solution u , for which the mapping Tu is a diffeomorphism, is the *mass balance* condition,

$$\int_{\Omega} f = \int_{\Omega^*} g. \quad (1.6)$$

For the standard Monge–Ampère equation, $c(x, y) = x \cdot y$, $Y = p$, $A = 0$, with $Tu = Du$, the classical solvability of the second boundary value problem for smooth densities and domains, under the mass balance condition (1.6), was proved by Delanöe, ($n = 2$), [3], Caffarelli [2] and Urbas [22], under the hypothesis that both domains, Ω and Ω^* , are uniformly convex, (in the sense that the principal curvatures of their boundaries are bounded from below by a positive constant). As already pointed out in [10, 11], (when Ω^* is a ball), (1.5) implies a nonlinear *oblique* boundary condition. A weaker interpretation of the boundary condition (1.5) arises through optimal

transportation, in which case Caffarelli [1] proved that the convexity of the target Ω^* suffices for local smoothness of solutions.

Interest in the general case was stimulated in the last decade through its application to regularity in optimal transportation. In the wake of earlier work on reflector design [27], a condition for local regularity called A3 was found by Ma, Trudinger and Wang in [15] and its degenerate form called A3w was introduced and used for global regularity in [20]. The latter was shown to be sharp by Loeper in [14], utilising the proof in [15] of the necessity of target c -convexity. We may write these conditions in the form:

A3 (A3w)

$$A_{ij}^{kl} \xi_i \xi_j \eta_k \eta_l := (D_{p_k p_l} A_{ij}) \xi_i \xi_j \eta_k \eta_l >, (\geq) 0,$$

for all $x, p \in \Omega \times \mathbb{R}^n$ such that $Y(x, p) \in \Omega^*$ and $\xi, \eta \in \mathbb{R}^n$ such that $\xi \cdot \eta = 0$. Using the notation,

$$c_{ij\dots,kl\dots} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \cdots \frac{\partial}{\partial y_k} \frac{\partial}{\partial y_l} \cdots c$$

and writing $[c^{i,j}]$ for the inverse of $c_{x,y} = [c_{i,j}]$, we also have

$$A_{ij}^{kl} = (c_{ij,pq} - c^{r,s} c_{ij,s} c_{r,pq}) c^{p,k} c^{q,l}, \quad (1.7)$$

in accordance with the original formulation in [15]. Note that from (1.7), we see that conditions A3 and A3w are symmetric in x and y , while the formulation in terms of x and p shows they are invariant under coordinate transformations in x , for fixed y .

For global regularity, we also need uniform convexity conditions on our domains, with respect to c , as also formulated in [20], namely we assume that Ω, Ω^* are respectively uniformly c -convex, c^* -convex with respect to each other in the sense that the images $c_y(\cdot, y)(\Omega), c_x(x, \cdot)(\Omega^*)$ are uniformly convex in \mathbb{R}^n for all $y \in \overline{\Omega}^*, x \in \overline{\Omega}$.

Theorem 1.1 *Let $c \in C^4(\overline{\Omega} \times \overline{\Omega}^*)$ be a cost function satisfying conditions A1, A2 and A3w in C^4 domains Ω, Ω^* which are mutually uniformly c -convex, c^* -convex with respect to each other. Suppose $f \in C^2(\overline{\Omega}), g \in C^2(\overline{\Omega}^*)$ are positive densities satisfying the mass balance condition (1.6). Then there exists an elliptic solution $u \in C^3(\overline{\Omega})$ of the second boundary value problem (1.3), (1.5), with Tu a diffeomorphism from $\overline{\Omega}$ to $\overline{\Omega}^*$, which is unique up to additive constants.*

The corresponding Monge–Kantorovich problem in optimal transportation is to find a measure preserving mapping $T_0 : \Omega \rightarrow \Omega^*$ which maximizes the cost functional

$$C(T) = \int_{\Omega} f(c(\cdot, T)) dx \quad (1.8)$$

among all measure preserving mappings T from Ω to Ω^* . A mapping $T : \Omega \rightarrow \Omega^*$ is called *measure preserving*, with respect to densities f and g , if it is Borel measurable and for any Borel set $E \subset \Omega^*$,

$$\int_{T^{-1}(E)} f = \int_E g. \quad (1.9)$$

The reader is referred to the expositions [16, 23, 25, 26] for further information about optimal transportation.

Corollary 1.2 *Under the hypotheses of Theorem 1, there exists a unique diffeomorphism $T = Y(\cdot, Du)$, $\in [C^2(\overline{\Omega})]^n$ maximizing the functional (1.8), where u is an elliptic solution of the second boundary value problem (1.3), (1.5).*

The solution u of (1.3), (1.5) is called a potential. From elliptic regularity theory it follows that if $c, \Omega, \Omega^*, f, g$ are C^∞ smooth, then the resultant potentials u and optimal mapping T are also. Note that we have followed the same sign conventions as in [20] but usually the cost functions and potentials are the negatives of those here and the optimal transportation problem is written equivalently as a minimization problem.

These results are proved in [20] under a further hypothesis, namely either the non degenerate condition A3 holds or a global barrier type condition is satisfied in Ω , (called c -boundedness in [17, 20]), or the matrix function A depends only on p . Such restrictions arise from the second derivative bounds in Section 3 of [20] and do not affect the main examples as presented there in Section 8. The last mentioned alternative sufficient condition follows from a duality argument in Section 3 of [20] which does not seem to extend to general dependence of A on (x, p) , (as envisaged in an earlier version of [20]). An alternative sub-solution condition for these second derivative bounds is also given in the recent paper [6]. In the next section we indicate the necessary modifications to obtain the full generality.

We remark also that the results and proofs in [20] were extended to the more general prescribed Jacobian equations in [18] and to parabolic equations in [9].

2 Second derivative bounds

Theorem 1.1 follows from a modification to the main second derivative estimate in [20], (also labelled Theorem 1.1). To facilitate the application of the method of continuity from [20], we need to consider more general equations than (1.3), namely we assume still that A is given by (1.4) but consider a general $B > 0$, $\in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. We also allow for a domain variation.

Theorem 2.1 *Let the matrix function A and domains Ω and Ω^* satisfy the same conditions as in Theorem 1.1 with the scalar function B as above and let Ω_t and Ω_t^* denote C^4 subdomains of Ω and Ω^* , which are also uniformly c and c^* -convex with respect to each other. Then any elliptic solution $u \in C^3(\overline{\Omega}_t)$ of (1.1), (1.5), for which the mapping Tu is a diffeomorphism from Ω_t to Ω_t^* , satisfies the a priori estimate*

$$|D^2u| \leq C, \quad (2.1)$$

where C depends on $c, \Omega_t, \Omega_t^*, B$ and $\sup_{\Omega_t} |u|$.

More specifically, with c , Ω and Ω^* fixed, the constant C depends on the C^2 norms and minima of B over compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^n$ together with the C^4 smoothness of the domains Ω_t and Ω_t^* and their uniform convexity constants. We also observe the simple gradient estimate

$$|Du| = |c_x(\cdot, Tu)| \leq \sup_{\Omega \times \Omega^*} |c_x|. \quad (2.2)$$

Proof From [20] it suffices to derive a global bound for D^2u in terms of its boundary trace as that is subsequently estimated in Section 4 of [20], using the obliqueness estimate from Sect. 2. First we note from [20] that the uniform c -convexity of Ω implies the existence of a C^2 defining function ϕ for Ω satisfying

$$\left[D_{ij}\phi - c^{l,k} c_{ij,l}(\cdot, Tu) D_k\phi \right] \xi_i \xi_j \geq \delta_0 |\xi|^2, \quad (2.3)$$

in a fixed neighbourhood \mathcal{N} of $\partial\Omega$, for all $\xi \in \mathbb{R}^n$ and for a fixed constant $\delta_0 > 0$. Noting also the formula,

$$D_{p_k} A_{ij}(\cdot, Du) = c^{l,k} c_{ij,l}(x, Tu),$$

we have from the global second derivative estimate, Theorem 3.1 in [20],

$$\sup_{\mathcal{N}} |D^2u| \leq C \left(1 + \sup_{\partial\mathcal{N}} |D^2u| \right), \quad (2.4)$$

with constant C depending on c , Ω , Ω^* , B and $\sup(|u| + |Du|)$. Consequently the global estimation is reduced to an interior estimate in a strictly contained subdomain $\Omega' = \Omega - \overline{\mathcal{N}}$.

Interior second order derivative estimates for solutions of Monge–Ampère type equations, under the weak condition A3w, are derived in [12] and [13]. We need to apply these in c -sections of u whose diameters will be controlled by the strict convexity results in [4] and [24]. First we note that the ellipticity condition (1.2) and the c -convexity of the domain Ω implies that u is c -convex in Ω [21], (see also [20] and [8]), that is for any $x_0 \in \Omega$ and $y_0 = Tu(x_0)$, we have

$$u(x) \geq u(x_0) + c(x, y_0) - c(x_0, y_0) \quad (2.5)$$

for all $x \in \Omega$. The c -section, $S_h(x_0)$ for $h > 0$ is then defined by

$$S_h(x_0) = \{x \in \Omega \mid u(x) < u(x_0) + c(x, y_0) - c(x_0, y_0) + h\}. \quad (2.6)$$

Writing $b = B(\cdot, u, Du)/|\det c_{x,y}(\cdot, Tu)|$, it follows since u is c -convex and Tu a diffeomorphism, that u is a potential for the corresponding Monge–Kantorovich problem with densities $f = b$ and $g = 1$. Furthermore there exist positive constants b_0 and b_1 , depending on B , $|\det c_{x,y}|$ and $\sup(|u| + |Du|)$ such that $b_0 \leq b \leq b_1$. We

now claim that, from [4] and [24], we have a strict convexity estimate, that is for any $R < \text{dist}(x_0, \partial\Omega_t)$, there exists $h > 0$, depending on c, Ω, Ω^* and b_0, b_1 , such that

$$S_h(x_0) \subset B_R(x_0) \quad (2.7)$$

where $B_R = B_R(x_0)$ denotes the ball of radius R and centre at x_0 . This follows in a standard way by contradiction for otherwise there would exist a sequence of potentials converging uniformly to a c -convex potential of a limiting problem with densities bounded away from 0 and ∞ which would not be strictly c -convex at x_0 , thereby contradicting the strict convexity assertions in [4] and [24]. We also note that a crucial hypotheses in these works is the uniform c -convexity of the target domain which is satisfied by hypothesis in our case. So that there is no confusion, we note also that *uniform* convexity in our terminology, as employed for example in [7], is designated *strong* in [4], (in accordance with usage in convexity theory where *uniform* can mean a uniform modulus of convexity).

To complete the proof of Theorem 2.1, we then note that for sufficiently small R there exists a smooth defining function ϕ satisfying (2.3) in B_R so that we can apply the interior second derivative estimate in [12] in $S_h(x_0)$ to obtain an estimate

$$\sup_{B_\rho} |D^2 u| \leq C, \quad (2.8)$$

for sufficiently small ρ depending on h and $\sup |D(u - c(\cdot, y_0))|$. By using a finite covering the estimate (2.8) is extended to Ω' and we conclude Theorem 2.1, with the asserted constant dependence, through replacement of Ω and Ω^* by Ω_t and Ω_t^* respectively. \square

We remark that a stronger result is proved in [4] than that used above, namely that the optimal mappings are strictly convex and Hölder continuous whereas in [24] only strict convexity and continuity is proved, (using an earlier version of [4] which assumes condition A3w without the orthogonality restriction on ξ and η). Note also that in two dimensions the strict convexity was already proved in [5] under a one-sided density bound as an extension of the classical result of Aleksandrov for the Monge–Ampère equation, (see for example [19]). However the argument in [5] also depends on a result established in [4]. We remark also that when the strong condition A3 holds, then the proof of the second derivative estimate is much simpler and already follows from [15]; see also [19].

From the estimates (2.1), (2.2), Theorem 1.1 follows immediately by using Theorem 2.1, (instead of Theorem 1.1 in [20]), in the method of continuity argument presented in Section 5 of [20].

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